Optimal Risk Management Using Options

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ABSTRACT

This article provides an analytical solution to the problem of an institution optimally managing the market risk of a given exposure by minimizing its Value-at-Risk using options. The optimal hedge consists of a position in a single option whose strike price is independent of the level of expense the institution is willing to incur for its hedging program. This optimal strike price depends on the distribution of the asset exposure, the horizon of the hedge, and the level of protection desired by the institution. Moreover, the costs associated with a suboptimal choice of exercise price are economically significant.

ONLY RECENTLY HAVE ACADEMICS begun to study the risk management practices of financial institutions and other corporations.[1] This is surprising given that for some time now, according to recent surveys (Smithson (1996a, 1996b)), the majority of firms have been applying modern financial techniques to managing some of their exposure to interest rates, equities, or exchange rates. One of the difficulties in analyzing these institutions' risk management programs is that their concept of risk is quite different from the standard measures implied by multifactor pricing models. Ceteris paribus, according to modern finance theory, it is cheaper for shareholders to diversify project risks on their own. Thus, a company's need to hedge either the systematic or unsystematic risk of its cash flows is limited.

However, there are several reasons why this standard argument may not hold true.[2] First, to avoid costly external financing, firms may need risk management programs to maintain their access to cheap capital—that is, internal funds (Froot et al. (1993) and Stulz (1990)). Second, in order to reduce the value of the government's implicit call option on the firm's assets

[1] See Allayannis and Ofek (1996), DeMarzo and Duffie (1995), Froot, Scharfstein, and Stein (1993), May (1995), Mian (1996), Smith and Stulz (1985), Stulz (1990), and Tufano (1996), among others, for a discussion of the underlying theory and empirics for why firms may have incentives to hedge, and, given these incentives, how firms implement the hedges.

[2] See Stulz (1997) for a general overview of institutions' risk management practices and incentives.

via taxes, risk management programs that lead to lower earnings volatility may be optimal (Smith and Stulz (1985)). Third, without some type of risk management at the institutional level, it may not be possible to disentangle business-related profits/losses from profits/losses associated with market exposures (DeMarzo and Duffie (1995)). Fourth, financial institutions facing risk-based capital requirements may find reducing risk to be cheaper than raising additional capital. Finally, risk management programs can reduce the costs of financial distress (Smith and Stulz (1985)).

Of course, these motivations for risk management are driven not by the magnitude of the firm's *market risk*, but rather by the magnitude of its *total risk*. More specifically, it is the probability and magnitude of potential losses that determine the desire to hedge, especially in the case of hedging motivated by the costs of external financing and financial distress. As a result of this different criteria for risk, the Value-at-Risk (VaR) concept has become the standard tool in the exploding area of risk measurement and management. In brief, VaR is an estimate of the probability and size of the potential loss to be expected over a given period. Although a growing number of approaches exist to answer the question of how to measure this VaR, academics and practitioners alike have been silent on the question of how to go about *managing* this risk.

We provide an analytical approach to optimal risk management in a framework that relies on two key assumptions. First, the institution's risk management criteria is VaR. Second, the institution's hedging strategy involves options, rather than forwards, futures, or swaps. The problem is to find a put option strategy that minimizes the VaR (given a maximal expenditure for hedging) by determining the optimal trade-off between the put options' ability to reduce the VaR level and the initial cost of these options. The solution is in the form of the put options' strike prices as a function of the underlying asset value, the mean and volatility of the asset's return, the risk-free rate, and the VaR hedging period. The analysis is performed in a Black–Scholes setting; therefore, it is better suited to the problem of hedging exposures to exchange rates, equities, or similarly distributed assets.

The main results can be summarized as follows. First, the optimal strategy involves a hedge position in a single option whose strike price is independent of the level of expense the institution is willing to incur for its hedge program. That is, given the fundamental parameters, the optimal option always has the same strike price.

Second, we are able to characterize the functional relation between the choice of the put option and the underlying parameters. The optimal strike price of this option is increasing in the asset's drift, decreasing in its volatility for most reasonable parameterizations, decreasing in the risk-free rate, and nonmonotonic in the maturity of the hedge. The distribution of the underlying asset exposure is the most important factor, and the optimal choice is very sensitive to the relative magnitude of the drift and diffusion of this exposure. Interestingly, the strike price is also increasing in the level of protection desired by the institution (i.e., the percentage of the distribution relevant for VaR); therefore, this choice is not innocuous.

Third, we show that the benefits of choosing the option optimally are economically significant. For example, using parameters that are typical for equity indexes, the VaR reduction using at-the-money options can be 45 percent less than the VaR reduction with an optimal hedge. Alternatively, using at-the-money options can require over 80 percent more in hedging expenditures to achieve the same VaR.

The paper is organized as follows. Section I describes the general problem and motivates the underlying assumptions. We provide two important preliminary results and build intuition via a graphical analysis of the problem of minimizing VaR using options. Section II presents the main theoretical analysis, including the solution to the VaR control problem and the comparative static results. We also illustrate these results in the context of a numerical example, and we quantify the benefits from the optimal choice of options. Section III concludes.

I. The General Optimization Problem

A. *Specification*

The starting point of our analysis is the classical hedging example in which an institution has an exposure to the price risk of an underlying asset, S_t, whose process is governed by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t, \tag{1}$$

where μ and σ are the drift and the volatility of the asset value, and z_t is a standard Brownian motion. This asset may be an exchange rate or a basket of exchange rates in the case of a multinational corporation, oil prices in the case of an energy provider, gold prices in the case of a mining company, etc. The only requirement is that this portfolio's return follows a geometric Brownian motion. As such, the analysis is better suited to an institution concerned with its exposure to commodity prices, equities, or exchange rates.

The institution hedges the asset's value using put options. Define the market price today (i.e., time t) of a τ-period put as $P_t = P(S_t, X, r, \tau, \sigma)$, where the strike price of the option equals X and the interest rate is r. For simplicity, we assume that all options are priced according to the Black–Scholes option pricing model:

$$P_t = Xe^{-r\tau}\Phi(d_1) - S_t\Phi(d_2), \tag{2}$$

where

$$d_1 = \frac{\ln(X/S) - \left(r - \dfrac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \tag{3}$$

$$d_2 = \frac{\ln(X/S) - \left(r + \dfrac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \tag{4}$$

and $\Phi(\cdot)$ is the cumulative normal distribution. A put option strategy consists of long positions, h_i, $i = 1,\ldots,n$, in n options with strike prices X_i, $i = 1,\ldots,n$. The total cost of the put option strategy, $\sum_{i=1}^{n} h_i P_{it}$, cannot exceed a given fixed threshold, C. Additionally, we assume that the exposure is never fully hedged, that is, $\sum_{i=1}^{n} h_i < 1$. In general, this constraint will not bind for reasonable levels of expenditures on hedging.[3]

Finally, the institution is concerned about its exposure to the asset over the next τ periods, and the relevant measure of risk is the position's VaR. Define $\text{VaR}_{t+\tau}$ as the dollar loss at the α percent level of the distribution on the institution's exposure relative to investing the time t value of the portfolio in the risk-free asset. This future value provides the natural benchmark because a riskless portfolio will thereby yield a VaR of zero. The VaR of a position translates to the statistical statement: "With $(1 - \alpha)$ percent confidence, the dollar loss in the future value of the cash flow in τ periods will not exceed \$$\text{VaR}_{t+\tau}$." To calculate this VaR, note first that the conditional distribution of the future value of the unhedged asset is lognormal:

$$\ln S_{t+\tau} \sim \text{N}[m, s^2], \tag{5}$$

where

$$m = \ln S_t + (\mu - \tfrac{1}{2}\sigma^2)\tau, \tag{6}$$

$$s = \sigma\sqrt{\tau}. \tag{7}$$

Consequently, the VaR of the unhedged position is

$$\text{VaR}_{t+\tau} = S_t \exp(r\tau) - S_t \exp(\theta(\alpha)), \tag{8}$$

where

$$\theta(\alpha) \equiv (\mu - \tfrac{1}{2}\sigma^2)\tau + c(\alpha)\sigma\sqrt{\tau}, \tag{9}$$

and $c(\cdot)$ is the cut-off point of the cumulative distribution of a standard normal. The second term in the VaR is simply the expected payoff of the asset at the α percent level.

Hedging with options affects the VaR in two ways: (i) the cost of the hedge reduces the future cash flows in every state of the world; and (ii) the payoffs of the options increase the cash flows when they finish in the money. It can

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[3] The more general case is solved and discussed in Ahn et al. (1997).

never be optimal to purchase options with exercise prices below the expected payoff at the α percent level because they will not affect the VaR. Consequently, for the purposes of the VaR calculation, we can assume that the put options finish in-the-money. The resulting VaR is

$$\text{VaR}_{t+\tau} = S_t \exp(r\tau)$$
$$- \left[\left(1 - \sum_{i=1}^{n} h_i \right) S_t \exp(\theta(\alpha)) + \sum_{i=1}^{n} h_i X_i - \left(\sum_{i=1}^{n} h_i P_{it} \right) \exp(r\tau) \right]. \quad (10)$$

The VaR depends on the α percent payoff of the partially unhedged exposure, the payoffs of the options, and the future value of the cost of the hedge. An equivalent interpretation of this last term is that the institution borrows the cost of the options and pays back the loan at expiration. Given this VaR, the institution's optimization problem is

$$\text{Min}_{h_i, X_i} \quad \text{VaR}_{t+\tau}$$

$$\text{subject to} \quad \sum_{i=1}^{n} h_i P_{it} \le C, \quad \sum_{i=1}^{n} h_i < 1, \quad h_i \ge 0 \quad \forall i. \quad (11)$$

The institution minimizes its VaR using long positions in put options, subject to a cost constraint on hedging and a constraint that the exposure be underhedged.

Before proceeding to a discussion of the solution to the optimization problem, there are three aspects of the specification that warrant further attention: (i) the choice of VaR as the measure of risk; (ii) the restriction of the set of hedging instruments to put options; and (iii) the expenditure constraint.

Although VaR is clearly not the result of some optimization over all possible risk management criteria, it may be a close first approximation. VaR and similar measures can be motivated via capital requirements in the case of financial institutions, or through some minimum level of funds necessary to perform business as usual in the case of other corporations. In any event, VaR is becoming an industry standard, and it provides an objective measure of risk.

Using forwards or futures to minimize the VaR of an institution's assets is straightforward and less interesting. Credit and basis risk aside, the VaR can be reduced to zero in the context of the optimization problem specified above. The key distinction between forwards and options is that the former instrument gives up some of the right tail of the distribution in order to reduce the left tail, but the latter works on the left tail only, albeit at an initial upfront cost. Moreover, though transacting forward is a common hedge methodology, recent surveys suggest that the use of options is also commonplace.[4] There are a number of reasons why institutions use options as a hedging vehicle. For example, the institution may be willing, or even have

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[4] See, for example, recent surveys reported by Smithson (1996a, 1996b).

the desire, to take the underlying asset exposure, leading to only a partial hedge of its cash flows. This will be true if the motivations for risk management are external financing costs, financial distress possibilities, managerial incentives, or tax optimization. Another reason is that institutional constraints, such as GAAP hedge accounting guidelines, might lead to forwards not being a viable alternative for some corporations. Even in the case where forwards are used to hedge some of the exposure, our optimization problem will still apply to hedging the VaR of the residual exposure.

The constraint on the cost of the hedge is motivated by issues of both practicality and liquidity. From a liquidity perspective, institutions may have a limited availability of funds for hedging, and raising additional funds may be difficult and costly. From a practical perspective, institutions have a limited appetite for costly VaR reduction. The solution to the optimization problem provides the VaR/cost frontier that shows the trade-off between hedging expenditures and VaR reduction for many expenditure levels, allowing the institution to select its desired point.

B. Preliminary Results

The optimization problem in equation (11) appears complex; however, there are two preliminary results that permit substantial simplification. First, the optimal hedging strategy consists of investment in a single put option. In other words, there is a single exercise price, from the set of available strike prices, that will provide the optimal trade-off between cost and VaR improvement. Intuitively, because the exposure is not fully hedged, the choice of option to hedge the remaining exposure at any level of current expenditure will always be the same. Alternatively, consider a strategy using puts with different exercise prices. Using a single put, whose exercise price is a weighted average of these exercise prices, generates the same payoff at the critical VaR percentile. However, the cost of this put is lower due to the convexity of put prices.[5]

Second, the constraint on hedging expenditures is binding. Given that the exposure is not fully hedged and that a single strike price is used, the last dollar spent provides the same cost/benefit trade-off as the first dollar. Consequently, the institution will always spend the maximum available.[6] Note, however, that this result does not imply that the institution will always want to spend more on hedging. Solving the optimization problem at a variety of cost levels yields a VaR/cost frontier—that is, the lowest level of VaR that can be achieved for each level of expenditure. The point on this frontier that the institution chooses may depend on liquidity constraints, capital requirements, or managerial preferences.

To build intuition for the optimal choice of strike price, consider the trade-off between cost and exposure. Put options with lower strike prices provide less protection, but they are cheaper; therefore, the institution can afford to

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[5] A proof of this result is available from the authors.

[6] There are some extreme parameter values for which it is not optimal to spend any money on hedging; however, if any amount of hedging is optimal, then the constraint will bind.

hedge a greater fraction of its exposure. In Section II, we provide analytical results for this problem, but it is worthwhile first to consider the problem from a graphical viewpoint.

Denote the future value of the hedged asset by $V_{t+\tau}$. If the put option finishes out-of-the-money, then the value is

$$(V_{t+\tau}|S_{t+\tau} \geq X) = S_{t+\tau} - hP_t \exp(r\tau) \tag{12}$$

and the distribution of this value is lognormal, shifted to the left by the future value of the cost of the option. If the put option finishes in-the-money, then the value is

$$(V_{t+\tau}|S_{t+\tau} < X) = (1-h)S_{t+\tau} + hX - hP_t \exp(r\tau) \tag{13}$$

and the distribution is again lognormal, due to the partially unhedged exposure, shifted to the right by the proceeds of the exercised option less the future value of the cost. Combining these distributions:

$$f(V_{t+\tau}) = \begin{cases} \dfrac{1}{\sqrt{2\pi}s(V_{t+\tau} + hP_t \exp(r\tau))} \exp\left[-\dfrac{1}{2}\left(\dfrac{\ln(V_{t+\tau} + hP_t \exp(r\tau)) - m}{s}\right)^2\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } V_{t+\tau} \geq X - hP_t \exp(r\tau) \\[2ex] \dfrac{1}{\sqrt{2\pi}s(V_{t+\tau} - hX + hP_t \exp(r\tau))} \\ \qquad \times \exp\left[-\dfrac{1}{2}\left(\dfrac{\ln(V_{t+\tau} - hX + hP_t \exp(r\tau)) - \ln(1-h) - m}{s}\right)^2\right] \\ \qquad\qquad \text{if } hX - hP_t \exp(r\tau) < V_{t+\tau} < X - hP_t \exp(r\tau) \\[2ex] 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if } V_{t+\tau} \leq hX - hP_t \exp(r\tau). \end{cases} \tag{14}$$

Figure 1 uses the above results to illustrate the trade-off between the strike price and the hedge ratio for three feasible combinations of exercise price and hedge ratio out of the continuum of possible choices. Panel A shows the distribution of the hedged payoffs, with the distribution of the unhedged payoff given by the solid line. The graph is based on the parameter values $S_t = 100$, $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$, and $\tau = 1$, and the hedging cost is fixed at 0.35 percent of the value of the underlying asset. For hedge ratios of 0.25, 0.5, and 0.75, the corresponding options are approximately 8 percent, 13 percent, and 15 percent out-of-the-money. As the hedge ratio increases, the strike price must decrease in order to maintain a fixed hedging cost. This trade-off is clear in the respective distributions. At a hedge ratio of 0.75 and a low exercise price, the distribution is almost fully hedged below a payoff of 80, but the protection declines rapidly thereafter. In contrast, at a hedge ratio of
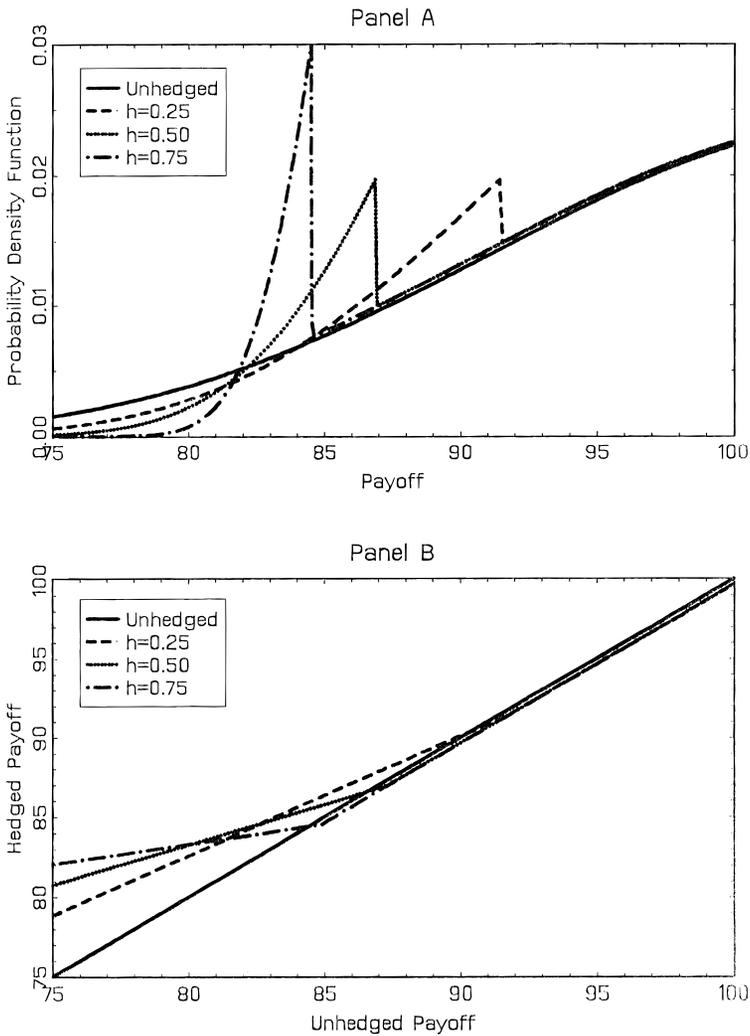
Figure 1. The exercise price/hedge ratio trade-off. The probability density function of the hedged value (Panel A) and the hedged value versus the unhedged value (Panel B) for different choices of hedge ratio, h, and exercise price, X, given a fixed hedging cost. The parameter values used are: the drift of the asset value, $\mu = 0.10$; the volatility of the asset value, $\sigma = 0.15$; the interest rate, $r = 0.05$; the horizon, $\tau = 1$; the asset value in time t, $S_t = 100$, and the cost threshold, $C = 0.35$.

0.25 and a higher exercise price, extreme events are more likely but a larger range of payoffs is hedged. The problem, of course, is to choose the option position to minimize the VaR at a given percentage level. Interestingly, the optimal exercise price (and hedge ratio) will depend on the particular percentage level chosen.

This dependence is illustrated in Figure 1, Panel B, which presents the value of the hedged position at maturity (i.e., $V_{t+\tau}$) versus the value of the underlying asset. The 45 degree line (the solid line) is the payoff assuming no hedging. The hedged payoffs for all the hedge ratios lie parallel to this line above their respective exercise prices because the option finishes out-of-the-money, and the institution loses the future value of the hedging expenditure. Below the exercise price, the slope of the hedged payoff depends on the hedge ratio—the higher the hedge ratio, the flatter the line. For a fully hedged position, the payoff would be horizontal below the exercise price.

From this graph it is relatively simple to calculate the α percent VaR for a given hedge ratio and exercise price pair, and thus to choose the best option. First, find the unhedged payoff that corresponds to the α percent level. The corresponding hedged payoff is the α percent payoff for the hedged distribution. Consequently, the hedge ratio (and exercise price) that provides the lowest VaR for a given percentage level corresponds to the highest payoff line for that corresponding underlying asset value. For small percentage levels (i.e., when the institution is concerned about larger potential losses that occur with a smaller probability), the optimal exercise price is lower and the hedge ratio is higher. For large percentage levels, it is optimal to use options with a higher exercise price and lower hedge ratios. At intermediate percentage levels, an intermediate exercise price is optimal.

II. Minimizing VaR with Options

A. Solution to the Minimization Problem

Given the results in Section I, the optimization problem faced by an institution as given in equation (11) can be rewritten as

$$\text{Min}_{h,X} \quad \text{VaR}_{t+\tau} = S_t \exp(r\tau) - [(1-h)S_t e^{\theta(\alpha)} + hX - hP_t \exp(r\tau)]$$

$$\text{subject to} \quad C = hP_t, \quad 0 \leq h < 1. \tag{15}$$

Substituting in the hedging cost constraint,

$$X^* = \text{argmin}_X S_t \exp(r\tau) - \left[\left(1 - \frac{C}{P_t}\right)S_t e^{\theta(\alpha)} + \frac{C}{P_t}X - C\exp(r\tau)\right]$$

$$= \text{argmax}_X C\left[\frac{X - S_t e^{\theta(\alpha)}}{P_t}\right] = \text{argmax}_X\left[\frac{X - S_t e^{\theta(\alpha)}}{P_t}\right]. \tag{16}$$

Some observations are in order. First, and perhaps most striking, the $\text{VaR}_{t+\tau}$ is an affine function of the hedging cost, C, and so it will not affect the choice of X. The optimal X is determined by the cash flow of the asset, and

the hedge ratio will adjust depending on the hedging costs.[7] This result also confirms the fact that the cost constraint is binding. The VaR is linear in the hedging expenditure, so each additional dollar generates the same reduction in VaR. There are no diminishing benefits to hedging.

Second, equation (16) shows that the minimization of $\text{VaR}_{t+\tau}$ is equivalent to the maximization of the ratio of the distance between the exercise price and the α percent level of the unhedged payoff, and the price of the put option. Loosely speaking, the objective function can be interpreted as the ratio of the benefit of hedging and the cost of hedging. Increasing the strike price of the option hedges a greater fraction of the distribution, but the option becomes more expensive.

The first-order condition for the maximization problem in equation (16) is

$$\frac{P_t - (X - S_t e^{\theta(\alpha)}) \dfrac{\partial P_t}{\partial X}}{P_t^2} = 0. \tag{17}$$

Hence, the solution X^* satisfies the nonlinear equation,

$$X^* - S_t e^{\theta(\alpha)} = \frac{P_t}{\dfrac{\partial P_t}{\partial X}} = X^* - S_t e^{r\tau} \frac{\Phi(d_2)}{\Phi(d_1)}$$

$$\Rightarrow \quad S_t e^{\theta(\alpha)} = S_t e^{r\tau} \frac{\Phi(d_2)}{\Phi(d_1)}. \tag{18}$$

One can interpret equation (18) in the following way. The strike price is chosen such that the α percent payoff of the unhedged position is equal to the *risk-neutral* expectation of the exposure conditional on the option being exercised.[8] In arriving at the above solution, we impose the budget constraint $C = hP(X)$. Consequently, $h^* = C/P(X^*)$; that is, the hedge ratio at the optimal exercise price is simply the cost divided by the value of the put option at that strike price. Though there is no closed-form solution for X^*, closed-form expressions are available for comparative statics using the implicit function theorem.

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[7] Recall that this result holds only if the resulting optimal hedge ratio is less than one. Ahn et al. (1997) discuss the less interesting case where hedging expenditures are sufficiently large so as to violate this assumption.

[8] Note that in equation (18), a necessary condition for the existence of solution X^* is that $\theta \le r\tau$ because $\Phi(d_2)/\Phi(d_1) < 1$. For most reasonable parameter values, this restriction will be satisfied. It essentially requires that the asset's drift not be too large relative to its diffusion. We thank Bruce Grundy for this observation.

B. Comparative Statics

Rewriting equation (18), the optimal choice of X satisfies

$$\Phi(d_1)\exp(\theta(\alpha) - r\tau) = \Phi(d_2). \tag{19}$$

Define $\beta = (\mu, \sigma, r, \tau)$. Since $X = X(S_t, \beta)$, using the implicit function theorem yields,

$$\frac{\partial X}{\partial \beta} = \frac{N(d_2)\dfrac{\partial d_2}{\partial \beta} - N(d_1)\dfrac{\Phi(d_2)}{\Phi(d_1)}\dfrac{\partial d_1}{\partial \beta} - \Phi(d_2)\dfrac{\partial(\theta - r\tau)}{\partial \beta}}{N(d_1)\dfrac{\Phi(d_2)}{\Phi(d_1)}\dfrac{\partial d_1}{\partial X} - N(d_2)\dfrac{\partial d_2}{\partial X}} \tag{20}$$

where

$$\frac{\partial d_1}{\partial X} = \frac{\partial d_2}{\partial X} = \frac{1}{\sigma\sqrt{\tau}X}, \tag{21}$$

and $N(\cdot)$ is the standard normal pdf. Taking the derivative of d_1 and d_2 with respect to each element of the parameter vector, β, yields the desired comparative statics results. The proofs of all these results can be found in Ahn et al. (1997).

The derivative of the optimal exercise price with respect to the drift of the underlying asset is

$$\frac{\partial X}{\partial \mu} = \frac{-\Phi(d_1)\Phi(d_2)\sigma X\tau^{3/2}}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)} \geq 0. \tag{22}$$

The effect of increasing the mean of the distribution is to increase the optimal strike price. The future distribution of the asset is shifted to the right relative to its current value; therefore, the optimal exercise price is also increased to preserve its relation relative to the α percent level of the unhedged payoff.

The derivative with respect to the underlying asset's volatility is

$$\frac{\partial X}{\partial \sigma} = \frac{X\sqrt{\tau}[N(d_1)\Phi(d_2)d_2 - N(d_2)\Phi(d_1)d_1 + \Phi(d_1)\Phi(d_2)(\sigma^2\tau - c(\alpha)\sigma\sqrt{\tau})]}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)}$$
$$\gtreqless 0, \tag{23}$$

which is of an indeterminate sign. As σ increases, the price of the put increases. Higher volatility also increases the dispersion of the distribution of the underlying asset. Consequently, the exercise price must decrease to pre-

serve its relation relative to the α percent level of the unhedged distribution, for reasonable values of α. Since both these effects work in the same direction, we might expect that as σ rises, the optimal strike price falls. For most parameterizations this is true. However, if $\alpha > 50$ percent, then the α percent level of the unhedged distribution is increasing in volatility and the unhedged $\mathrm{VaR}_{t+\tau}$ is decreasing in volatility. For a sufficiently high α this effect can offset the cost effect, and the optimal exercise price will be increasing in volatility.

The derivative with respect to the risk-free rate is

$$\frac{\partial X}{\partial r} = \frac{X[N(d_1)\Phi(d_2)\tau - N(d_2)\Phi(d_1)\tau + \Phi(d_1)\Phi(d_2)\sigma\tau^{3/2}]}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)} \leq 0. \tag{24}$$

As the interest rate increases, the optimal strike price decreases. The optimal strike price falls because of the corresponding fall in the cost of the put. However, because the effect on the cost is small and there is no effect on the distribution of the underlying asset, the overall effect of interest rate changes is small.

The derivative of the optimal exercise price with respect to the hedging horizon is

$$\frac{\partial X}{\partial \tau} = \frac{X\left[N(d_1)\Phi(d_2)\gamma_1 - N(d_2)\Phi(d_1)\gamma_2 - 2\sigma\tau\Phi(d_1)\Phi(d_2)\left(\mu - \frac{1}{2}\sigma^2 - r + \frac{c(\alpha)\sigma}{2\sqrt{\tau}}\right)\right]}{2\sqrt{\tau}[N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)]}$$
$$\gtreqless 0, \tag{25}$$

where

$$\gamma_1 = \sigma d_2 + 2r\sqrt{\tau}, \tag{26}$$

$$\gamma_2 = \sigma d_1 + 2r\sqrt{\tau}. \tag{27}$$

The horizon over which the partial option hedge takes place can have a large, yet nonmonotonic, effect on the optimal level of moneyness of the option. On the one hand, as the horizon increases, the positive drift in the asset's return dominates, and the strike price rises to reflect the shift in the distribution of the asset's value away from its current value. On the other hand, the volatility of the asset increases with the horizon, and the distribution gets more disperse, leading to lower optimal exercise prices. As the horizon gets very long, the former effect dominates, and the strike price increases. For shorter horizons, the volatility effect dominates, and the strike price decreases. In general, this reversal will always occur (as long as the drift is positive); however, its point of inflection depends on the underlying parameter values themselves.

A final interesting question to consider is how the optimal strike price changes as a function of the institution's desired protection level—that is, the tail percent of the distribution used to calculate the VaR. Again, using the implicit function theorem,

$$\frac{\partial X}{\partial \alpha} = \frac{-\Phi(d_1)\Phi(d_2)X\sigma^2\tau\Phi^{-1'}(\alpha)}{N(d_1)\Phi(d_2) - N(d_2)\Phi(d_1)} \geq 0, \tag{28}$$

where $\Phi^{-1'}(\cdot)$ is the derivative of inverse function of the cumulative normal distribution. Both the denominator and the numerator are negative, so the optimal exercise price is increasing in α. This result is consistent with the intuition provided at the end of Section I.B. Since this level is a choice variable of the institution, one could imagine using these results to help the institution trade off the choice of options against the amount they are willing to pay and the desired level of protection.

C. An Illustration of Optimal Hedging

In order to illustrate some of the above results, and to quantify the benefits associated with optimal hedging, we turn to a numerical example. Throughout this example we use the parameter values $S_t = 100$, $\mu = 0.10$, $\sigma = 0.15$, $r = 0.05$, $\tau = 1$, and $\alpha = 2.5$ percent.

For these parameter values, the optimal X^* is $87.59, and the institution should purchase options 12.41 percent out-of-the-money. If no hedging takes place, the VaR is $23.68; however, by purchasing $0.35 worth of put options, the VaR is reduced to $21.15. As shown above, the VaR is linear in the hedging expenditure, and, for this example, every $0.10 of put options reduces the VaR by $0.72. The institution can then trade off its VaR reduction versus the cost of this reduction. One key point is that the optimal level of the moneyness of the option is invariant to these costs.

It is worthwhile to quantify the benefit of a judicious (i.e., optimal) choice of exercise price relative to a suboptimal choice. To do this, we compare the VaR and cost of a hedged position using options with various exercise prices. We answer two related questions. First, given a certain cost allocation for hedging, how does the VaR using the optimal exercise price options compare to the VaR using other exercise prices? Second, given a targeted VaR level, how does the cost of implementation differ across different choices of exercise prices? The results are presented in Figure 2.

Figure 2, Panel A, plots the VaR as a function of the exercise price. Each line represents a certain level of expenditure on the options hedge. The VaR declines as the cost allocated for the hedge increases, but it is also sensitive to the exercise price of the put option. The VaR of the position is minimized for options with an exercise price of $87.59 for any expenditure level because the optimal exercise price is independent of the cost. For a cost of $0.35, the VaR is minimized at $21.15, a reduction of $2.53 relative to the unhedged
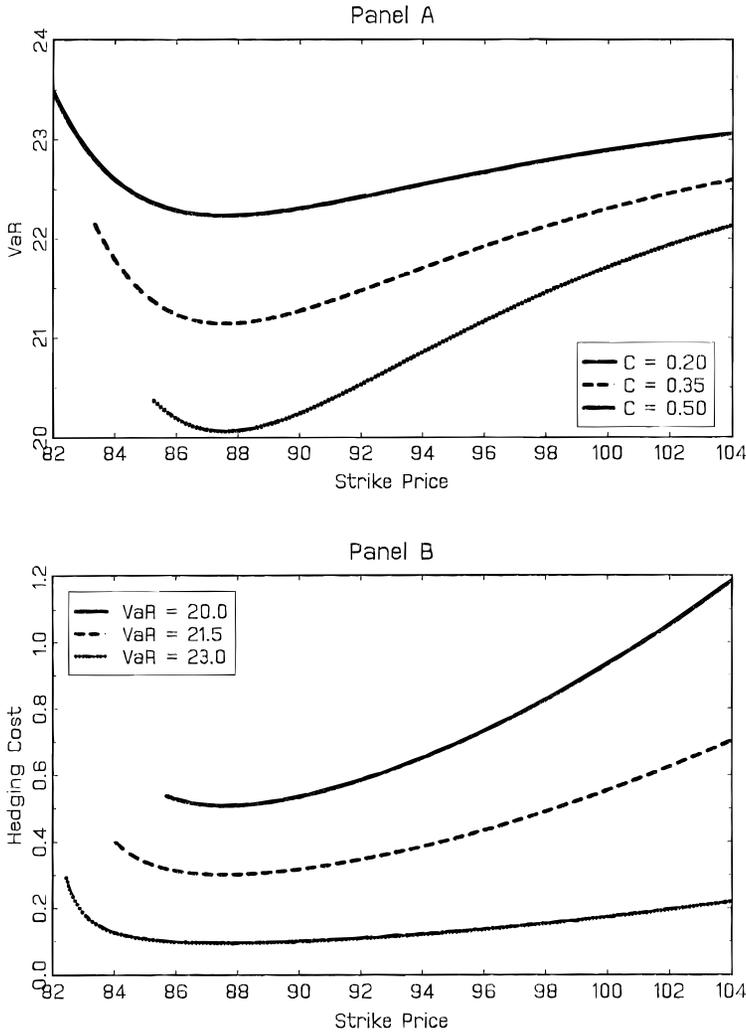
Panel A

Panel B

Figure 2. The cost of suboptimal hedging. VaR for a fixed hedging cost, C (Panel A), and hedging cost for a fixed VaR (Panel B) as a function of the exercise price. The parameter values used are: the asset value at time t, $S_t = 100$; the drift of the asset value, $\mu = 0.10$; the volatility of the asset value, $\sigma = 0.15$; the interest rate, $r = 0.05$; the horizon, $\tau = 1$; and the Var tail, $\alpha = 2.5\%$.

level. Increasing the exercise price to \$100 (i.e., using an at-the-money option) yields a VaR of \$22.30, a reduction of only \$1.38. In other words, an economically meaningful 45 percent of the hedging benefit is lost by using a suboptimal exercise price. Similar magnitudes are evident for all expenditure levels.

Figure 2, Panel B, addresses the same issue from a slightly different perspective, showing the cost of hedging across various exercise prices holding the targeted VaR level fixed. The expenditure is minimized at the optimal
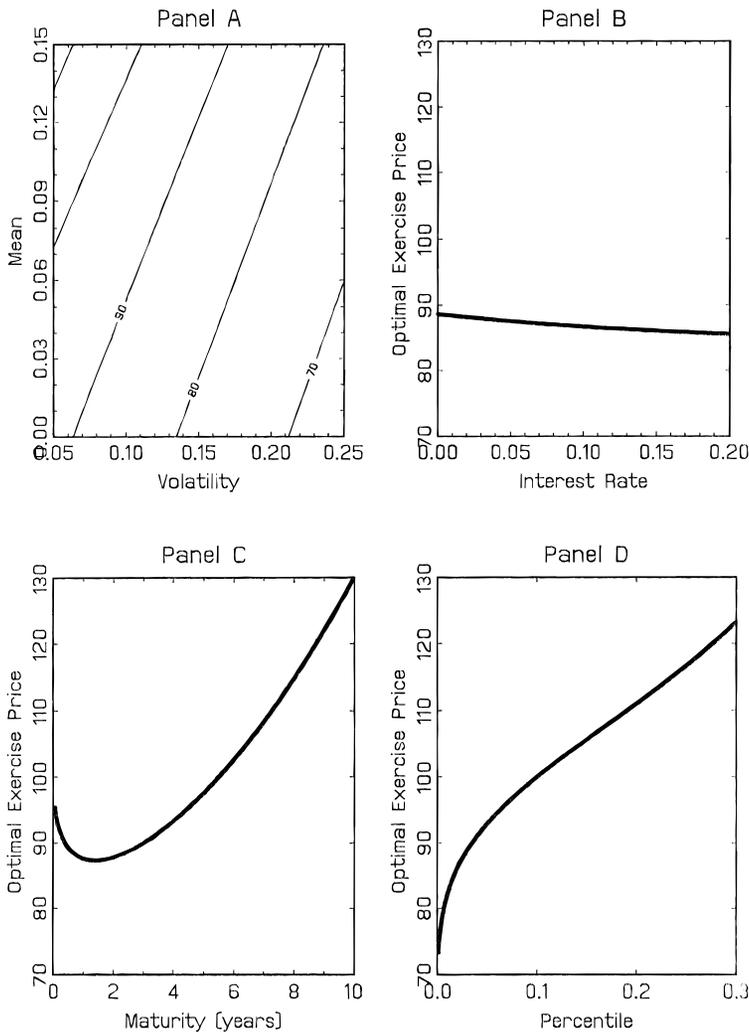
Figure 3. The determinants of the optimal exercise price. The effects of changing the mean and volatility (Panel A), the interest rate (Panel B), the horizon (Panel C), and the percentile (Panel D) on the optimal exercise price. The underlying parameter values are: the asset value at time t, $S_t = 100$; the drift of the asset value, $\mu = 0.10$; the volatility of the asset value, $\sigma = 0.15$; the interest rate, $r = 0.05$; the horizon, $\tau = 1$; and the Var tail, $\alpha = 2.5\%$.

exercise price, and deviations from this optimal point again yield economically meaningful losses. For example, if a VaR of $21.50 is desired, implementing it using at-the-money options would cost $0.55, but at the optimal exercise price the cost would be $0.30. The institution must increase its expenditure by more than 80 percent to achieve the same risk reduction using a suboptimal exercise price.

Figure 3 illustrates how X^* varies with the underlying parameters. Panel A provides a contour graph that illustrates the optimal strike price as a

function of the distribution of the future value of the underlying asset (i.e., the parameters μ and σ). The exercise price is positively related to the drift and negatively related to the volatility. As a result, it depends on the relative level of the two parameters. For example, for the pairs ($\mu = 0.04$, $\sigma = 0.08$), ($\mu = 0.065$, $\sigma = 0.10$), and ($\mu = 0.09$, $\sigma = 0.115$), the optimal exercise price is 92, or eight percent out-of-the money. As the drift and volatility vary from these values, the optimal strike price can vary quite dramatically, from deep out-of-the-money to in-the-money.

In contrast, Panel B shows that, though the strike price is decreasing in the risk-free rate, the effect is of second order. For example, increasing r from five percent to 20 percent causes the optimal level of moneyness to fall from 12.5 percent to only 14.4 percent out-of-the money. Note that the y-axis scale is identical for the remaining figures.

Figure 3, Panel C, shows that the horizon over which the partial option hedge takes place has a large and nonmonotonic effect on the optimal level of moneyness of the option. As the horizon increases to one year, the optimal strike price decreases from six percent to 12 percent out-of-the-money. Between a one-year and a two-year horizon the relation between horizon and strike price reverses. It is at this point that the mean effect begins to dominate the volatility effect. For a horizon of seven years, the optimal exercise price is 10 percent in-the-money.

The final determinant of the exercise price is the level of desired protection. Panel D graphs the optimal strike price against the desired α percent level of protection. Obviously, as additional protection is desired, more and more of the distribution of the asset needs to be hedged against, and the strike price rises. However, the figure shows that this relation between the strike price and α is highly nonlinear. Consequently, an institution or regulatory body setting capital requirements should take these results into account when deciding on the appropriate tail probability. For example, going from $\alpha = 2.5$ percent to $\alpha = 10$ percent increases the optimal exercise price of the option from \$87.59 to \$100.00 and significantly decreases the fraction of the exposure that is hedged.

III. Conclusion

This paper provides a formal analysis of optimal risk control using options in a simplified framework in which an institution wishes to minimize its VaR. The complication arises when considering a menu of possible pairs of exercise prices and hedge ratios given a level of expenditure, since such different choices imply different levels of hedged VaR. We find that the optimal strike price is independent of the level of cost; therefore, the cost/VaR frontier is linear. That is, given the parameters governing the distribution of asset returns, and the desired confidence level, an institution faces the choice of increasing the position in an optimal exercise price option, thereby reducing its VaR. Interestingly, the choice of optimal exercise price is sensitive to the desired confidence level.

There are several natural extensions of our analysis to nonnormal distributions, mean reverting processes, fixed-income securities, etc. The most natural extension, however, is to multiple asset exposures. Examples include an exporter/importer with exposure to various exchange rates, a pension fund manager with exposure to equity and bond markets, or an energy company with exposure to the cost of various energy sources. The optimization can then be extended to the question of the optimal choice of a *menu* of options on the different underlying exposures, taking into consideration a richer set of parameters, namely the correlations among assets (which may provide a natural hedge). However, because a portfolio of options is generally more expensive than an option on a portfolio, the risk management problem is best addressed by approaching the over-the-counter options market and constructing an option on the compound position. In doing so, the analysis falls back within the realm of our model, as long as the distributional assumptions hold. One might argue that the recent explosion in the use of over-the-counter basket options is related to this issue.

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